

University of Augsburg
Faculty of Theoretical Physics I



Studies on the adiabatic theorem in classical mechanics

Bachelor Thesis

by

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1 Introduction

Every physics student learned in the thermodynamics lecture, what an adiabatic process in thermodynamics is. During an adiabatic process in thermodynamics the system exchanges no heat Q with the environment.

$$\delta Q = 0 \quad \Rightarrow \quad \delta U = \delta W \quad (1.1)$$

These adiabatic processes can be in experiments approximated by quick processes, because then the process duration is so short, that only a small amount of heat can be exchanged. In this bachelor thesis will only adiabatic processes in Hamilton mechanics be examined. In Hamilton mechanics the heat concept does not exist, so there the definition of an adiabatic process must be different.

The situation in Hamilton mechanics is, that one particle is caught in a potential and this potential transform from its initial form to a different one. The process is called adiabatic, if the change of the potential per period of the particle is very small. Therefore every adiabatic process in Hamilton mechanics is very slow.

Now arises the question, how these two different definitions of an adiabatic process are connected with each other? In thermodynamics you work with the assumption, that the system is always in its equilibrium state. That is only possible, when the movements of the particles in the system are much faster than the thermodynamic process. Or formulated in the other way around, that in the time for a typical movement of a particle the change of the system caused by the thermodynamic process is small. This sound very familiar with the definition of an adiabatic process in Hamilton mechanics.

When in Hamilton mechanics the potential alters its form, then it performs work on the particle. Because the single particle has no contact with other particles, the performed work is equal to the energy gain of the particle like in 1.1. So the connection between energy and work is equal in both cases of an adiabatic process. This short consideration should has shown the familiarity of the two definitions for adiabatic processes.

In most cases of adiabatic processes in Hamilton mechanics the phasespace is an invariant value. This effect is often used to calculated the energy of the particle during and after the process without solving the Hamilton equations. This is often taught in lectures, but also applied in current research like in this papers [5]. In this paper a theoretical process is designed to cool one particle in a potential. In this thesis the interest is not focused to use this effect, but instead to create adiabatic processes, which change the phasespace of the particle in the potential.

2 Analytical part

2.1 Definition of an adiabatic invariant

We examine a one dimensional mechanical system with an Hamiltonian

$$H = H(q(t), p(t), \lambda(t)) \quad (2.1)$$

during the time interval $[0, t_{\text{final}}]$. In case the variable $\lambda(t)$ is constant the system performs a periodic motion with a constant energy E and cycle duration T . In mechanics the adiabatic limes is the case, when the change of λ during each period is small:

$$T \left| \frac{d\lambda(t)}{dt} \right| \ll |\lambda(t)| \quad \forall t \in [0, t_{\text{final}}] \quad (2.2)$$

When $\lambda(0)$ and $\lambda(t_{\text{final}})$ are constant and not equal, the ending time t_{final} is much greater than one cycle duration T .

$$T \ll t_{\text{final}} \quad (2.3)$$

An adiabatic invariant I is a value of the system, which changes slowly enough in the adiabatic limes.

$$\frac{dI}{dt} = O(\dot{\lambda}^{1+\delta}) \quad \delta > 0 \quad (2.4)$$

If I changes with a higher order of $\dot{\lambda}$ than 1, then the total change of I can be estimated as zero in the adiabatic limes.

$$\begin{aligned} |I(t_{\text{final}}) - I(0)| &= \left| \int_0^{t_{\text{final}}} \frac{dI(t)}{dt} dt \right| \leq \int_0^{t_{\text{final}}} \left| \frac{dI(t)}{dt} \right| dt \\ &\leq \int_0^{t_{\text{final}}} \max_{t' \in [0, t_{\text{final}}]} \left| \frac{dI(t')}{dt'} \right| dt = \max_{t' \in [0, t_{\text{final}}]} \left| \frac{dI(t')}{dt'} \right| t_{\text{final}} \\ &\stackrel{2.4}{=} O(\dot{\lambda}^{1+\delta}) t_{\text{final}} \stackrel{2.7}{=} O(\dot{\lambda}^\delta) \xrightarrow{\dot{\lambda} \rightarrow 0} 0 \end{aligned} \quad (2.5)$$

In this calculation t_{final} is replaced with the help of the average $\bar{\lambda}$.

$$\bar{\lambda} = \frac{1}{t_{\text{final}}} \int_0^{t_{\text{final}}} \dot{\lambda}(t) dt = \frac{\lambda(t_{\text{final}}) - \lambda(0)}{t_{\text{final}}} \quad (2.6)$$

$$\Rightarrow t_{\text{final}} = \frac{\lambda(t_{\text{final}}) - \lambda(0)}{\bar{\lambda}} = O(\dot{\lambda}^{-1}) \quad (2.7)$$

In case $\lambda(t_{\text{final}})$ is equal to $\lambda(0)$, then the adiabatic process must be split-up into two processes to avoid a division by zero. The inner time border t_{middle} must be chosen with $\lambda(t_{\text{middle}})$ not equal to $\lambda(0)$ and this is choice is possible, if $\lambda(t)$ is not constant.

2.2 Theorem: Phasespace is an adiabatic invariant

In some textbooks you find the theorem, that phasespace is an adiabatic invariant. Examples for such textbooks are [4], [3] and [1]. The phasespace A is the integration of the momentum p over the location q for one period of the system.

$$A = \oint pdq \quad (2.8)$$

So it is necessary to have a periodic system, to calculate the phasespace. In this section the Hamiltonian $H = H(q, p, \lambda)$ from section 2.1 is used. This system is periodic in case of a constant λ .

Now a typical way to proof the adiabatic invariance of the phasespace is shown. But first we need some equations to use during the proof. The functions of locations $q(t)$ and $p(t)$, you normal only get by solving the Hamilton's equations

$$\dot{q} = \left. \frac{\partial H}{\partial p} \right|_{q,\lambda} \quad \dot{p} = - \left. \frac{\partial H}{\partial q} \right|_{p,\lambda} \quad (2.9)$$

for explicit start conditions. When you split the cycle in suitable sections, then in each section of the cycle the momentum p can explicit be calculated by the variables q , E and λ .

$$p = p(q, E, \lambda) \quad (2.10)$$

Next we need to derive $E = H(q, p, \lambda)$ and $E = H(q, p(q, E, \lambda), \lambda)$ with respect to t , E and λ .

$$\frac{dE}{dt} = \frac{dH(q, p, \lambda)}{dt} = \left. \frac{\partial H}{\partial q} \right|_{p,\lambda} \dot{q} + \left. \frac{\partial H}{\partial p} \right|_{q,\lambda} \dot{p} + \left. \frac{\partial H}{\partial \lambda} \right|_{q,p} \dot{\lambda} \stackrel{2.9}{=} \left. \frac{\partial H}{\partial \lambda} \right|_{q,p} \dot{\lambda} \quad (2.11)$$

$$1 = \left. \frac{\partial E}{\partial E} \right|_{q,\lambda} = \left. \frac{\partial H(q, p(q, E, \lambda), \lambda)}{\partial E} \right|_{q,\lambda} = \left. \frac{\partial H}{\partial p} \right|_{q,\lambda} \left. \frac{\partial p}{\partial E} \right|_{q,\lambda} \quad (2.12)$$

$$\Rightarrow \left. \frac{\partial p}{\partial E} \right|_{q,\lambda} = \left(\left. \frac{\partial H}{\partial p} \right|_{q,\lambda} \right)^{-1} \quad (2.13)$$

$$0 = \left. \frac{\partial E}{\partial \lambda} \right|_{q,E} = \left. \frac{\partial H(q, p(q, E, \lambda), \lambda)}{\partial E} \right|_{q,E} = \left. \frac{\partial H}{\partial p} \right|_{q,\lambda} \left. \frac{\partial p}{\partial \lambda} \right|_{q,E} + \left. \frac{\partial H}{\partial \lambda} \right|_{q,p} \quad (2.14)$$

$$\Rightarrow - \left. \frac{\partial p}{\partial \lambda} \right|_{q,E} = \left. \frac{\partial H}{\partial \lambda} \right|_{q,p} \left(\left. \frac{\partial H}{\partial p} \right|_{q,\lambda} \right)^{-1} \quad (2.15)$$

2 Analytical part

Equipped with these equations the proof can be done:

$$\frac{d}{dt} \oint p(q, E, \lambda) dq = \left(\frac{dE}{dt} \frac{\partial}{\partial E} \Big|_{q, \lambda} + \frac{d\lambda}{dt} \frac{\partial}{\partial \lambda} \Big|_{q, E} \right) \oint p(q, E, \lambda) dq \quad (2.16)$$

$$= \oint \left(\frac{\partial p}{\partial E} \Big|_{q, \lambda} \frac{dE}{dt} + \frac{\partial p}{\partial \lambda} \Big|_{q, E} \dot{\lambda} \right) dq \quad (2.17)$$

$$\stackrel{2.11}{=} \oint \left(\frac{\partial p}{\partial E} \Big|_{q, \lambda} \frac{\partial H}{\partial \lambda} \Big|_{q, p} \dot{\lambda} + \frac{\partial p}{\partial \lambda} \Big|_{q, E} \dot{\lambda} \right) dq \quad (2.18)$$

$$\stackrel{2.13}{=} \oint \left(\left(\frac{\partial H}{\partial p} \Big|_{q, \lambda} \right)^{-1} \frac{\partial H}{\partial \lambda} \Big|_{q, p} \dot{\lambda} + \frac{\partial p}{\partial \lambda} \Big|_{q, E} \dot{\lambda} \right) dq \quad (2.19)$$

$$\stackrel{2.15}{=} \oint \left(- \frac{\partial p}{\partial \lambda} \Big|_{q, E} \dot{\lambda} + \frac{\partial p}{\partial \lambda} \Big|_{q, E} \dot{\lambda} \right) dq = 0 \quad (2.20)$$

This looks like an elegant way to proof the adiabatic invariance of the phasespace, but in this way it is wrong.

First of all it surprises, that \dot{A} is 0 and not $O(\dot{\lambda}^{1+\delta})$. So the phasespace would not only be an adiabatic invariant, it also would be an overall constant no matter how fast the change of λ would be. There exist simple counterexamples, which proof that the phasespace is not an overall constant.

Now it is time to find the root of the problem. While λ is constant the system performs a periodic movement, but in the adiabatic limes λ is not constant, so the system does not perform a periodic process. There are two ways to interpret the closed integral during an adiabatic process.

The first way is to take the integral with a fixed λ . In this case the energy E of the system would remain constant during the integral. So the equation 2.18 is wrong, because $\frac{dE}{dt}$ is 0 and equation 2.11 does only apply for the correct solution of the Hamiltonian not on an altered one. During the rest of this bachelor thesis the integral will be interpreted in this way.

The other way is to integrate from the start point to an end point, which is similar to the start point. For example the start point is $q = 0$ and $p = p_0$, so the end point could be $q = 0$ and $p = p_0 + \Delta p$. In this case it is allowed to use equation 2.11, but the integral would not be closed and so you must take the start and end point into consideration. So in equation 2.16 an extra term of $\frac{dq}{dt} \frac{\partial}{\partial q} \Big|_{E, \lambda}$ is needed on the right side and in equation 2.17 extra terms of the start and end points appear.

A few years ago C. G. Wells and S. T. C. Siklos presented in their scientific paper [6]. They also come up with an own way to proof the adiabatic invariance

of the phasespace, which lead to two interesting results. They defined an average phasespace J of all starting points with the same initial energy. The first result is, that the difference between the average and an arbitrary phasespace converge to zero in the adiabatic limes.

$$|J(t) - A(t)| = O(\dot{\lambda}) \quad t \in [0, t_{\text{final}}] \quad (2.21)$$

The next result is, that J changes with order in $\dot{\lambda}$ and $\ddot{\lambda}$.

$$\frac{dJ}{dt} = O(\ddot{\lambda}, \dot{\lambda}^2) \quad (2.22)$$

So the phasespace is an adiabatic invariant, if $\ddot{\lambda}$ gets with a higher order of $\dot{\lambda}$ then 1 in the adiabatic limes smaller.

$$\ddot{\lambda} = O(\dot{\lambda}^{1+\delta}) \quad \Rightarrow \quad |A(t_{\text{final}}) - A(0)| \leq \begin{cases} O(\dot{\lambda}) & \text{if } \delta > 1 \\ O(\dot{\lambda}^\delta) & \text{if } \delta \leq 1 \end{cases} \quad (2.23)$$

That the adiabatic invariance of the phasespace depends on $\ddot{\lambda}$ is the big difference between this proof and any other you find in textbooks.

In the following chapters there will be examples of different adiabatic Hamiltonians. The intention behind these examples is to find one, which changes his phasespace in the adiabatic limes. One tool to use in this task, is a not vanishing $\dot{\lambda}$. That would point out, that the different proofs of the adiabatic invariance of the phasespace, which can be found in textbooks, are wrong.

2.3 Used Hamiltonians

The Hamiltonians in the rest of this bachelor thesis describe one classic mechanical and punctual particle, which is caught in a convex potential. The potential depends only on the location q and a two dimensional time depending parameter $\lambda(t) = (\lambda_1(t), \lambda_2(t))^T$. When λ is constant the particle performs periodic motion and phasespace can be calculated. So the Hamiltonian has all traits like the one in chapter 2.1 and in case of a very small $\dot{\lambda}$ it is an adiabatic process. For simplicity the mass of the particle is set to 1.

$$H(q, p, \lambda) = \frac{p^2}{2} + V(q, \lambda) \quad (2.24)$$

As a potential only two different forms are later used. First V_1 , which is quadratic in the location q , and second V_2 , which is biquadratic in q , are used and both have a one dimensional parameter $\omega(q, \lambda)$.

$$V_1(q, t) = \frac{1}{2}\omega^2(q, \lambda)q^2 \quad (2.25)$$

$$V_2(q, t) = \frac{1}{4}\omega^4(q, \lambda)q^4 \quad (2.26)$$

2 Analytical part

$\omega(q, \lambda)$ is a very simple function. It returns $\lambda_1(t)$, if q is positive, and $\lambda_2(t)$ else. Later λ_1 and λ_2 will be just referred to as ω_r and ω_l , because these names are very intuitive as they describe the left and the right branch of the potentials.

$$\omega(q, t) = \begin{cases} \omega_r(t) = \lambda_1(t) & \text{if } x > 0 \\ \omega_l(t) = \lambda_2(t) & \text{if } x \leq 0 \end{cases} \quad (2.27)$$

For the change of momentum in the numerical calculation the force $F(q, t)$ is needed.

$$F_1(q, t) = -\frac{\partial V_1(q, t)}{\partial q} = -\omega^2(q, t)q - q^2\omega(q, t)\frac{\partial\omega(q, t)}{\partial q} \quad (2.28)$$

In the cases ($q < 0$) and ($q > 0$) $\frac{\partial\omega(q, t)}{\partial q}$ is zero, because ω is in each case constant. For the last case $q \rightarrow 0$ it can be shown with help of the difference quotient, that $-\frac{\partial V_1(q, t)}{\partial q}$ is there converging to zero, besides $\frac{\partial\omega(q, t)}{\partial q}$ is diverging. For the mathematical proof you have to differ in six cases and it can be found in the attachment (5.1). For the biquadratic potential it is the same and the forces simplifies to:

$$F_1(q, t) = -\omega^2(q, t)q \quad (2.29)$$

$$F_2(q, t) = -\omega^4(q, t)q^3 \quad (2.30)$$

To be able to compare the different potentials better, $\omega(q, t = 0)$ is a fixed constant called ω_0 .

$$\omega(q, t = 0) = \omega_l(t = 0) = \omega_r(t = 0) = \omega_0 \quad (2.31)$$

During a time τ ω_r and ω_l evolve to a constant value called ω_τ and after that they remain constant.

$$\omega(q, t \geq \tau) = \omega_l(t \geq \tau) = \omega_r(t \geq \tau) = \omega_\tau \quad (2.32)$$

To simplify it more ω_τ is one of two different values. The first value is $2\omega_0$ and the second is $\frac{1}{2}\omega_0$. In case ω_τ is $2\omega_0$, then $\omega_l(t)$ and $\omega_r(t)$ are monotonously increasing functions. In the other case they are monotonously decreasing functions. An example for the first case would be:

$$\omega(q, t) = \begin{cases} \omega_0 & \text{if } t = 0 \\ \omega_0(1 + \frac{t}{\tau}) & \text{if } 0 < t < \tau \text{ and } x > 0 \\ \omega_0(1 + (\frac{t}{\tau})^2) & \text{if } 0 < t < \tau \text{ and } x \leq 0 \\ 2\omega_0 & \text{if } t \geq \tau \end{cases} \quad (2.33)$$

In the adiabatic limes τ will steadily be increased, so $\left|\frac{\partial\omega(q, t)}{\partial t}\right|$ converge steadily to zero.

$$T \ll \tau \Rightarrow T \left|\frac{\partial\omega(q, t)}{\partial t}\right| \ll \omega(q, t) \quad (2.34)$$

2.4 Conservation of energy through special $\omega(q, t)$

The intention behind creating a $\omega(q, t)$ is to change the phasespace in the adiabatic limes. The approach in this thesis is to form $\omega(q, t)$ that way, that for a particle with special starting conditions its energy keeps constant, besides $\omega_1(t)$ and $\omega_r(t)$ change during the process. In that case the phasespace of this special particle would change even in the adiabatic limes. The way to do this, is that, when the particle is on the right side ($q > 0$), $\omega_r(t)$ must remain constant and $\omega_1(t)$ can alternate. Is the particle on the left side the roles of $\omega_r(t)$ and $\omega_1(t)$ are switched.

$$\begin{aligned} \text{if } q \leq 0 & \quad \frac{\partial \omega_r(q,t)}{\partial t} \neq 0 \quad \text{and} \quad \frac{\partial \omega_1(q,t)}{\partial t} = 0 \\ \text{if } q > 0 & \quad \frac{\partial \omega_1(q,t)}{\partial t} \neq 0 \quad \text{and} \quad \frac{\partial \omega_r(q,t)}{\partial t} = 0 \end{aligned} \quad (2.35)$$

In order to get a simple $\omega(q, t)$, the starting conditions of the special particle are $q(t = 0) = 0$ and $p(t = 0) = p_0 > 0$. For these starting conditions the particle spends exactly one half period on the right side and after that one half period on the left side and so on.

2.4.1 linear increase of $\omega_1(q, t)$ and $\omega_r(q, t)$ in their non-constant times

Now a $\omega(q, t)$ will be constructed, in which $\omega_1(t)$ and $\omega_r(t)$ have the constant derivation $\frac{2}{\tau}$ in their non-constant times. For this the interval $[0, \tau]$ is split into smaller intervals. These smaller intervals are the time in which the particle is either on the right or the left side of the potential. So this smaller intervals are exact one half period long. The tricky part is, that because of the increasing $\omega(q, t)$ the following interval is shorter then the prior one. The period duration $T(E, \omega)$ can be calculated by the formula

$$T(E, \omega) = \left. \frac{\partial A}{\partial E} \right|_{\omega}. \quad (2.36)$$

The derivation of this formula can be found in the attachment(5.2). For the quadratic and the biquadratic potential you get:

$$\text{quadratic potential : } T(\omega) = \frac{2\pi}{\omega} \quad (2.37)$$

$$\text{quadratic potential : } T(E, \omega) = \frac{3}{\omega E^{\frac{1}{4}}} \int_{-1}^1 \sqrt{1 - y^4} dy \quad (2.38)$$

2 Analytical part

With the formula for the period and the initial energy E_0 of the particle, which remain constant for the whole process, the first interval can be calculated and the results for $\omega_1(t)$ and $\omega_r(t)$ are:

$$t \in (0, t_0] \quad t_0 = \frac{1}{2}T(E_0, \omega_0) \quad \omega_1(t) = \omega_0(1 + \frac{2}{\tau}t) \quad \omega_r(t) = \omega_0 \quad (2.39)$$

In the second interval $\omega_1(t)$ and $\omega_r(t)$ are steadily continued, but now $\omega_1(t)$ is constant and $\omega_r(t)$ is increasing. Because the particle is now on the left side, the interval length will be calculated with $\omega_1(t_0)$.

$$\begin{aligned} t \in (t_0, t_1] & \quad \omega_1(t) = \omega_0(1 + \frac{2}{\tau}t_0) \\ t_1 = t_0 + \frac{1}{2}T(E_0, \omega_1(t_0)) & \quad \omega_r(t) = \omega_0(1 + \frac{2}{\tau}(t - t_0)) \end{aligned} \quad (2.40)$$

In the same way the $\omega_1(t)$ and $\omega_r(t)$ in the next two intervals can be found:

$$\begin{aligned} t \in (t_1, t_2] & \quad \omega_1(t) = \omega_0(1 + \frac{2}{\tau}(t - t_1 + t_0)) \\ t_2 = t_1 + \frac{1}{2}T(E_0, \omega_r(t_1)) & \quad \omega_r(t) = \omega_0(1 + \frac{2}{\tau}(t_1 - t_0)) \end{aligned} \quad (2.41)$$

$$\begin{aligned} t \in (t_2, t_3] & \quad \omega_1(t) = \omega_0(1 + \frac{2}{\tau}(t_2 - t_1 + t_0)) \\ t_3 = t_2 + \frac{1}{2}T(E_0, \omega_1(t_2)) & \quad \omega_r(t) = \omega_0(1 + \frac{2}{\tau}(t - t_2 + t_1 - t_0)) \end{aligned} \quad (2.42)$$

Now you should see the system, how to create the following intervals and the functions for $\omega_1(t)$ and $\omega_r(t)$. Important to consider is whether n is even oder odd.

$$\begin{aligned} \text{n even} \quad t \in (t_n, t_{n+1}] & \quad \omega_1(t) = \omega_0(1 + \frac{2}{\tau} \sum_{i=0}^n (-1)^i t_i) \\ t_{n+1} = t_n + \frac{1}{2}T(E, \omega_1(t_n)) & \quad \omega_r(t) = \omega_0(1 + \frac{2}{\tau}(t - \sum_{i=0}^n (-1)^i t_i)) \end{aligned} \quad (2.43)$$

$$\begin{aligned} \text{n odd} \quad t \in (t_n, t_{n+1}] & \quad \omega_1(t) = \omega_0(1 + \frac{2}{\tau}(t + \sum_{i=0}^n (-1)^i t_i)) \\ t_{n+1} = t_n + \frac{1}{2}T(E, \omega_r(t_n)) & \quad \omega_r(t) = \omega_0(1 - \frac{2}{\tau} \sum_{i=0}^n (-1)^i t_i) \end{aligned} \quad (2.44)$$

This definition is not exactly compatible with the requirements. The problem is, that $\omega_1(\tau)$ is slightly higher then $2\omega_0$ and $\omega_r(\tau)$ lower then $2\omega_0$. To solve this problem two corrections are needed. The first one is to stop $\omega_1(t)$ from increasing, when it reached the $2\omega_0$ mark and the second one is to extend the time until $\omega_r(t)$ also reached $2\omega_0$. An example with $\tau = T$ of the corrected $\omega(q, t)$ can be seen in figure 1.

2.4 Conservation of energy through special $\omega(q, t)$

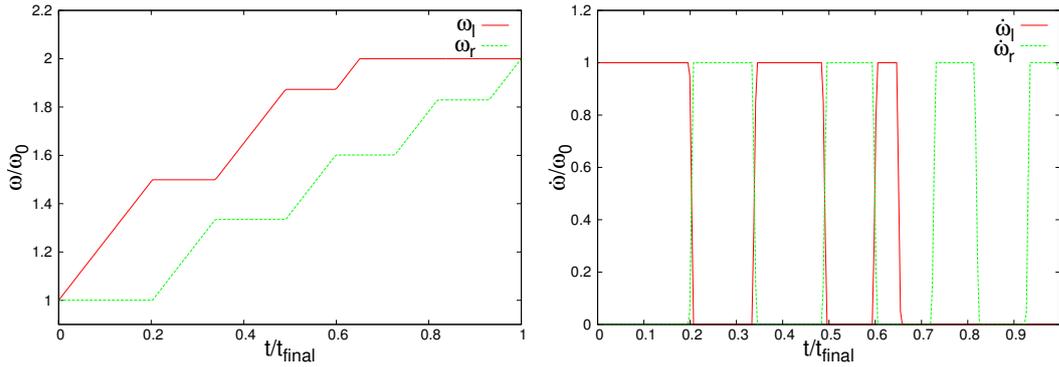


Figure 1: $\omega_l(t)$ and $\omega_r(t)$ (left) and their derivations with respect to t (right) for $\tau = 2T$

2.4.2 Construction of more $\omega(q, t)$

With the help of the new found $\omega(q, t)$ more $\omega(q, t)$ are constructed in this section. To the $\omega(q, t)$ from the chapter 2.4.1 will be referred as $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ and the new will still be named $\omega(q, t)$. For the construction also a function $f(x)$, which is defined on the interval $[0, 1]$, is required. In chapter 2.3 is a distinction of two cases for $\omega(q, t)$. In the first case the end value of $\omega(q, t)$ is $2\omega_0$ and $\omega_l(t)$ and $\omega_r(t)$ are monotonous increasing. To create this first case the function $f(x)$ must also be monotonous increasing and also it must have these boundary values:

$$f(0) = 1 \quad f(1) = 2 \quad (2.45)$$

In the second case $\omega_l(t)$ and $\omega_r(t)$ are monotonous decreasing and the end value of both is $\frac{1}{2}\omega_0$. Analogical he function $f(x)$ is monotonous decreasing and its boundary values are:

$$f(0) = 1 \quad f(1) = \frac{1}{2} \quad (2.46)$$

The new $\omega(q, t)$ is simply:

$$\omega(q, t) = \omega_0 f\left(\frac{\omega_{(\text{pos})}^{(\text{lin})}(q, t)}{\omega_0} - 1\right) \quad (2.47)$$

In order to keep the correct side constant t_i of $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ have to be adjusted.

$$t_0 = \frac{1}{2}T(E_0, \omega_0) \quad t_{i+1} = t_i + \frac{1}{2}T\left(E_0, \omega_0 f\left(\frac{\omega_{(\text{pos})}^{(\text{lin})}{}_{r,l}(t_i)}{\omega_0} - 1\right)\right) \quad (2.48)$$

Whether $\omega_l(t_i)$ or $\omega_r(t_i)$ is taken, is done analogical to the normal $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$. During this thesis used examples for $f(x)$ are:

2 Analytical part

$\omega_\tau = 2\omega_0$		$\omega_\tau = \frac{1}{2}\omega_0$	
$\omega_{(\text{pos})}^{(\text{lin})}(q, t):$	$f(x) = 1 + x$	$\omega_{(\text{neg})}^{(\text{lin})}(q, t):$	$f(x) = 1 - \frac{1}{2}x$
$\omega_{(\text{pos})}^{(\text{quad})}(q, t):$	$f(x) = 1 + x^2$	$\omega_{(\text{neg})}^{(\text{quad})}(q, t):$	$f(x) = 1 - \frac{1}{2}x^2$

The resulting $\omega(q, t)$ can be seen in the figures 2, 3 and 4.

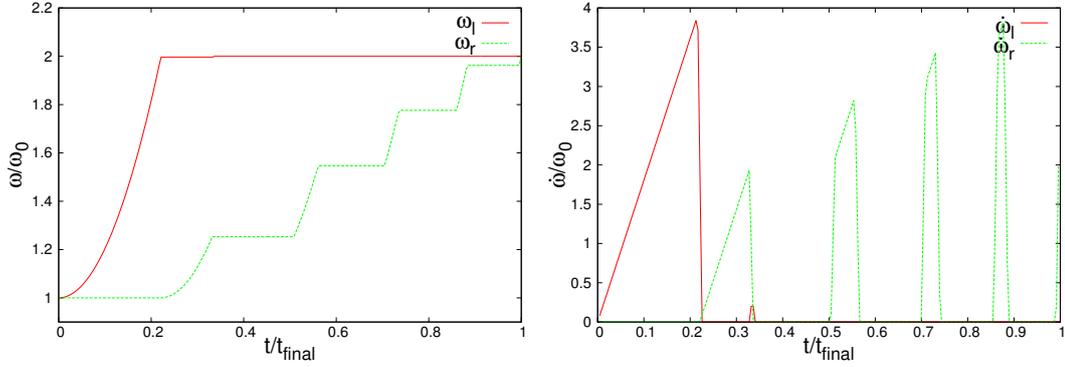


Figure 2: $\omega_1(t)$ and $\omega_r(t)$ of $\omega_{(\text{pos})}^{(\text{quad})}(q, t)$ (left) and their derivations with respect to t (right) for $\tau = 2T$

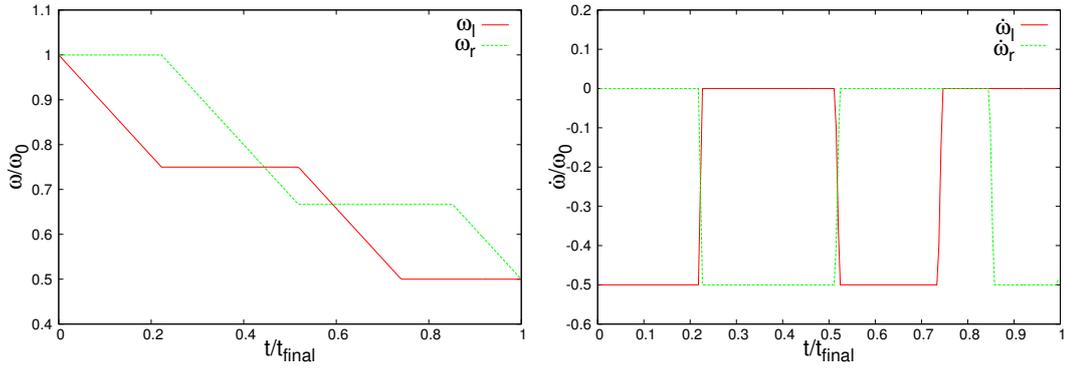


Figure 3: $\omega_1(t)$ and $\omega_r(t)$ of $\omega_{(\text{neg})}^{(\text{lin})}(q, t)$ (left) and their derivations with respect to t (right) for $\tau = 2T$

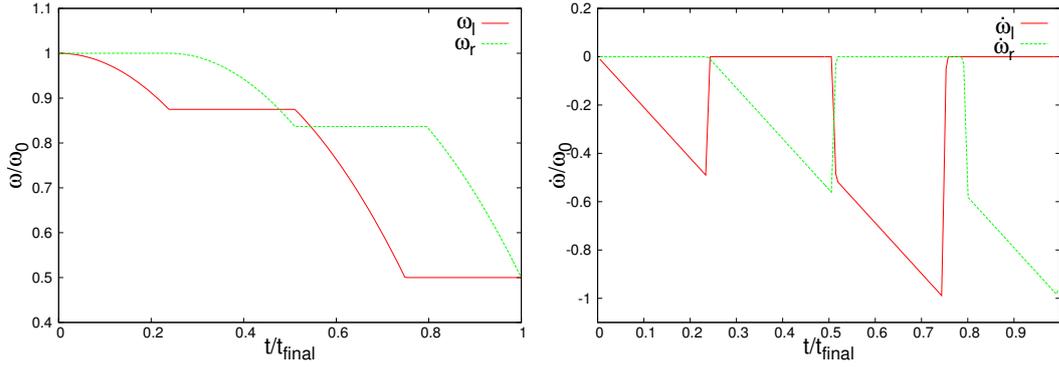


Figure 4: $\omega_l(t)$ and $\omega_r(t)$ of $\omega_{(\text{neg})}^{(\text{quad})}(q, t)$ (left) and their derivations with respect to t (right) for $\tau = 2T$

3 Numerical part

3.1 Simulation

In the prior sections the Hamiltonians $H(q, p, \lambda)$ are constructed, now it is time to solve them. For the particle with the start conditions $q(t=0) = 0$ and $p(t=0) = p_0 > 0$ the solution is simple, because its energy E remains for the whole process constant. For particles with different start conditions the problems can not be solved analytical, so a numerical approach must be done.

As a computer language C/C++ is used and every value is calculated with a *double* accuracy, except for the integer values. The differential equations are solved by the *Leapfrog method* and *Neri 4th order method*. These are two explicit symplectic integrators. Symplectic integrators solve a slightly perturbed Hamiltonian and their formulas are taken from [2]. The great advantage of symplectic integrates compared to non-symplectic integrates is, that the energy for a non time depending Hamiltonian remains constant. Because the phasespace is the wanted value, the accuracy of the energy is crucial, hence these two values are directly connected with each other.

$$\text{quadratic potential : } A = \pi q_{\text{max}} p_{\text{max}} = 2\pi \frac{E}{\omega} \quad (3.1)$$

$$\text{biquadratic potential : } A = \frac{E^{\frac{3}{4}}}{\omega} 4 \int_{-1}^1 \sqrt{1-y^4} dy \quad (3.2)$$

If you use a non-symplectic integrator, then the error of the total energy grows secularly. E.g. solving the normal harmonic oscillator with the *classical Runge Kutta method*, then the total Energy increases steadily over time and does not

3 Numerical part

remain constant and if you use *implicit Euler method*, then the total energy decreases over time. Higher order symplectic integrators could be used to improve the accuracy of the numerical integration, how you could construct them is described here [7]. The increment for the numerical integration dt is chosen as $\frac{T}{1000}$, because then the solutions of the *leapfrog method* and the *Neri 4th order method* differ at most in the fourth digit after the dot.

In the following sections the numerical results are shown for the four particles with the start conditions

$$q_1 = 0 \quad p_1 = p_0 \quad (3.3)$$

$$q_2 = 0 \quad p_2 = -p_0 \quad (3.4)$$

$$q_3 = q_0 \quad p_3 = 0 \quad (3.5)$$

$$q_4 = -q_0 \quad p_4 = 0 \quad , \quad (3.6)$$

whereat q_0 and p_0 are chosen, that the particles have the same initial energy E_0 . Besides the phasespace of these four particles an average phasespace \bar{A} of 1000 different particles is also calculated. These 1000 particles have all the initial energy E_0 , but each got a different start condition. When you keep ω equal to ω_0 , then the system is periodic with a period duration T_0 . The initial conditions q_i and p_i are calculated with Hamiltonian $H(q(t), p(t), \omega_0)$ of the periodic system:

$$H(q(t), p(t), \omega_0) : \quad q_i = q(t = \frac{i}{1000}T_0) \quad p_i = p(t = \frac{i}{1000}T_0) \quad 1 \leq i \leq 1000 \quad (3.7)$$

To avoid systematic errors, because of this uniform splitting of the interval $(0, T_0]$. The results of some samples are compared to an average phasespace of 1000 random particles:

$$H(q(t), p(t), \omega_0) : \quad q_i = q(t = n_i T_0) \quad p_i = p(t = n_i T_0) \quad 1 \leq i \leq 1000 \quad (3.8)$$

n_i is a random number out of the interval $(0, 1]$. With the use of 1000 particles the results of the uniform and the random splitting is within a good accuracy equal. E.g. when you use only 100 particles, then there is some perturbation between the two results.

3.2 Numerical results for the quadratic potential

First arises the question, how to describe the adiabatic limes? The adiabatic limes is defined as $|\dot{\lambda}| \rightarrow 0$. Our λ is connected to the time τ :

$$\max_{t \in [0, t_{\text{final}}]} (|\dot{\lambda}(t)|) \propto \frac{1}{\tau} \quad (3.9)$$

3.2 Numerical results for the quadratic potential

To find the adiabatic limes the final phasespace A_τ is calculated as a function of τ . The adiabatic limes is reached, if A_τ practical does not change over some powers of ten. This is not a mathematical rule, but it is very reasonable. In case of increasing $\omega(q, t)$ the diagrams in figure 5 show the results in which the τ -scale is logarithmic. The different phasespaces come from the different initial conditions of the particles:

$$A_1 : q_{\text{start}} = 0 \quad p_{\text{start}} = p_0 \quad (3.10)$$

$$A_2 : q_{\text{start}} = 0 \quad p_{\text{start}} = -p_0 \quad (3.11)$$

$$A_3 : q_{\text{start}} = q_0 \quad p_{\text{start}} = 0 \quad (3.12)$$

$$A_4 : q_{\text{start}} = -q_0 \quad p_{\text{start}} = 0 \quad (3.13)$$

$$\bar{A} : \text{average of 1000 different particles} \quad (3.14)$$

In figure 5 you see, that A_1 is constant $\frac{1}{2}A_0$, because the potential was

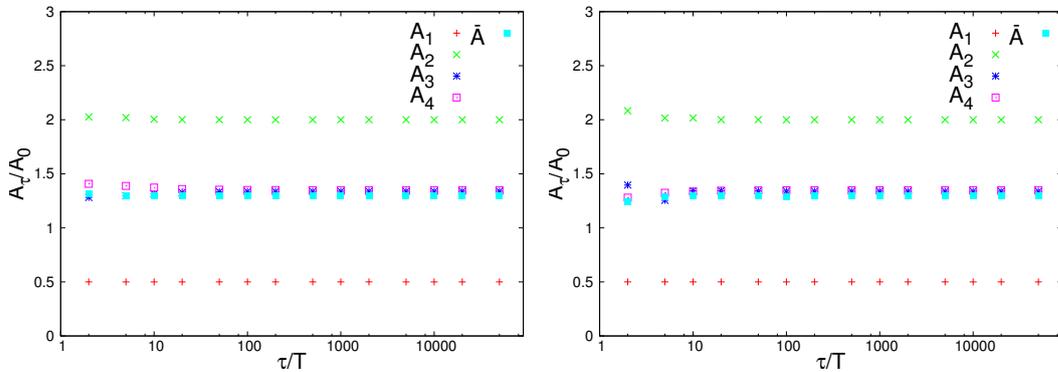


Figure 5: Adiabatic limes of $\omega_{(\text{plus})}^{(\text{lin})}(q, t)$ (left) and $\omega_{(\text{plus})}^{(\text{quad})}(q, t)$ (right)

constructed to remain the energy of particle 1 constant. So p_{max} of particle 1 is in the end of the process identical to the start, but q_{max} is reduced to its half, because ω doubled its value over the process. This leads with $A = \pi q_{\text{max}} p_{\text{max}}$ to an halved phasespace. The other four phasespace remain also quiet constant apart from slight perturbations at small τ . Over the last 10 of 14 points every phasespace practical does not change, so every phasespace should have reached its value in the adiabatic limes within a very good accuracy. That A_2 has doubled its phasespace, can also easily be explained. Because for big τ particle 2 is always on the upcoming branch of $\omega(q, t)$ and so its energy gets doubled by $\omega_r(t)$ and also gets doubled by $\omega_1(t)$. With the formula $A = 2\pi \frac{E}{\omega}$ the phasespace gets doubled in sum. The other three phasespaces A_3 , A_4 and \bar{A} can not be explained that simple. Because A_1 and A_2 are the most extreme values, so the three other values must lay between A_1 and A_2 .

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The main intention of this thesis is to show, that the adiabatic limes is no sufficient reason for the adiabatic invariance of the phasespace. In the assumption, that here the adiabatic limes is reached, these examples show, that the phasespace of the single particles A_1 , A_2 , A_3 and A_4 and an average phasespace \bar{A} does change over an adiabatic process.

Now it is time to check, whether this example is congruent with the formulas 2.21 and 2.22 found by Wells and Siklos. The first formula 2.21 implies, that the phasespace of two particles with the same initial phasespace differ just in order of $\dot{\lambda}$ for all times, that includes the ending time t_{final} at which the phasespace reaches its value A_τ :

$$|A_1(t) - A_2(t)| = |A_1(t) - J(t) + J(t) - A_2(t)| \leq \quad (3.15)$$

$$\leq |J(t) - A_1(t)| + |J(t) - A_2(t)| \stackrel{2.21}{=} O(\dot{\lambda}) = O\left(\frac{1}{\tau}\right) \quad (3.16)$$

In this example the difference of the end phasespace of A_1 and A_2 is in approximation constant $\frac{3}{2}A_0$ and does not changes in order of $\frac{1}{\tau}$. So this formula $|J(t) - A(t)| = O(\dot{\lambda})$ is here wrong and so forth it is not valid in general.

For the second equation 2.22 the second derivation of λ with respect to t is needed. To visualise the first and the second derivation every 10th numerical step the values of $\omega_1(t)$ and $\omega_r(t)$ got saved and by the difference quotients

$$\dot{\omega}(t) \approx \frac{\omega(t + dt) - \omega(t - dt)}{2dt}, \quad (3.17)$$

$$\ddot{\omega}(t) \approx \frac{\omega(t + dt) - 2\omega(t) + \omega(t - dt)}{dt^2} \quad (3.18)$$

the derivations are approximated. As an example $\omega_1(t)$ and $\omega_r(t)$ for $\tau = 10T$ can be seen in figure 6, the associated first derivation in figure 7 and the second derivation is in figure 8. When you look at the $\ddot{\omega}_1(t)$ and $\ddot{\omega}_r(t)$ in figure 8, then you see, that they run vertical up and down at some points. These are *Dirac delta functions* and the finite width is just an result of the difference quotient approximation. So $\max(|\ddot{\lambda}|)$ is an infinite number, in which case the formula $\frac{dJ}{dt} = O(\dot{\lambda}^2, \ddot{\lambda})$ (2.22) makes no restriction for the growth of $J(t)$, which is approximated by \bar{A} . Thus this example does not interfere with the second formula 2.22 found by Wells and Siklos. The next step is to create an smoothed $\omega(q, t)$, so its $\ddot{\omega}_1(t)$ and $\ddot{\omega}_r(t)$ get smaller by $\frac{1}{\tau}$.

3.2 Numerical results for the quadratic potential

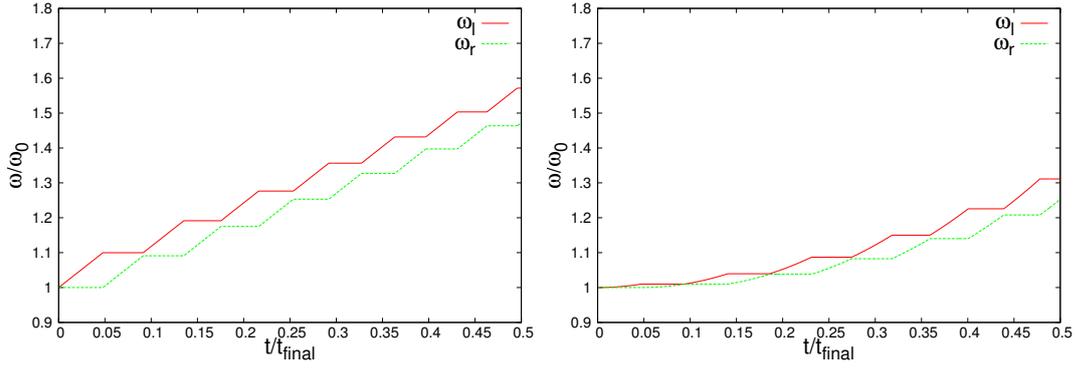


Figure 6: $\omega_1(t)$ and $\omega_r(t)$ of $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ (left) and $\omega_{(\text{pos})}^{(\text{quad})}(q, t)$ (right) for $\tau = 10T$

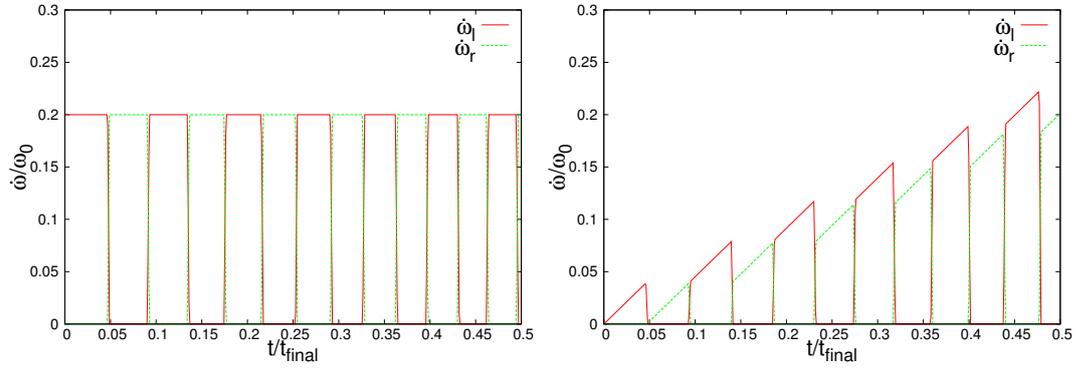


Figure 7: $\dot{\omega}_1(t)$ and $\dot{\omega}_r(t)$ of $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ (left) and $\omega_{(\text{pos})}^{(\text{quad})}(q, t)$ (right) for $\tau = 10T$

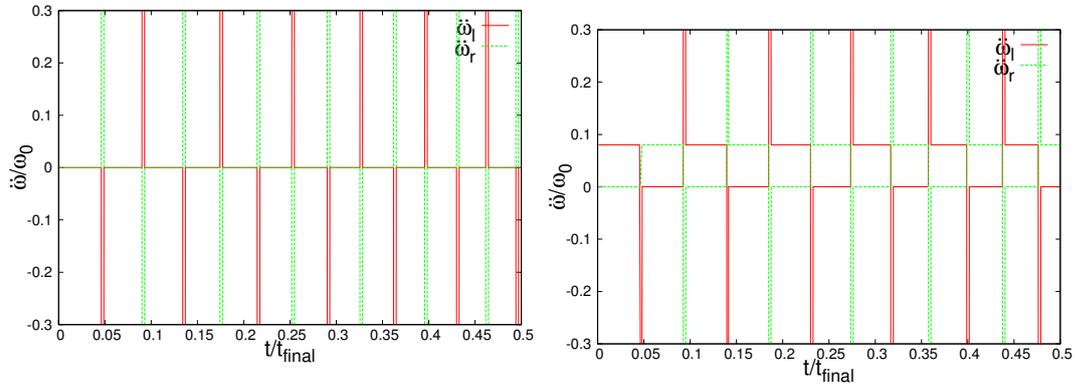


Figure 8: $\ddot{\omega}_1(t)$ and $\ddot{\omega}_r(t)$ of $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ (left) and $\omega_{(\text{pos})}^{(\text{quad})}(q, t)$ (right) for $\tau = 10T$

3.3 Smoothed $\omega(q, t)$ for the quadratic potential

The basic idea again is, that the Energy of the particle with initial conditions $q_1(t=0) = 0$ and $p_1(t=0) = p_0 > 0$ remains constant, while $\omega_1(t)$ and $\omega_r(t)$ increase from ω_0 to $2\omega_0$. In order to avoid *Dirac delta functions* in $\dot{\omega}_1(t)$ and $\dot{\omega}_r(t)$ $\omega_1(t)$ and $\omega_r(t)$ have to be continuous functions. When the examined particle is at $q_1(t) = 0$, then both $\dot{\omega}_1(t)$ and $\dot{\omega}_r(t)$ have to be zero.

An easy way to implement such a $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$, is to take this:

$$\dot{\omega}_1(t) = \begin{cases} 0 & \text{if } q_1(t) \leq 0 \\ \frac{p_0 - |p_1(t)|}{p_0\tau} & \text{if } q_1(t) > 0 \end{cases} \quad \dot{\omega}_r(t) = \begin{cases} \frac{p_0 - |p_1(t)|}{p_0\tau} & \text{if } q_1(t) \leq 0 \\ 0 & \text{if } q_1(t) > 0 \end{cases} \quad (3.19)$$

$$|p_1(t)| = \sqrt{2E_0 - \omega(q_1(t), t)^2 q_1(t)^2} \quad (3.20)$$

$q_1(t)$ and $p_1(t)$ are the location and the momentum of the particle with the initial conditions $q_1(t=0) = 0$, $p_1(t=0) = p_0$ and $E_0 = \frac{p_0^2}{2}$. It is important that the resulting $\omega(q, t)$ is constructed by the motion of this special particle and does not depend on the actual particle in the potential. For the special particle the motion $q_1(t)$ is quite simple and does only depend on the time t . Thence the resulting $\omega_1(t)$ and $\omega_r(t)$ do only depend on the time t . Now $\omega_1(t)$ and $\omega_r(t)$ and their first and second derivations with respect to the time can be seen here: figure 9 and 10

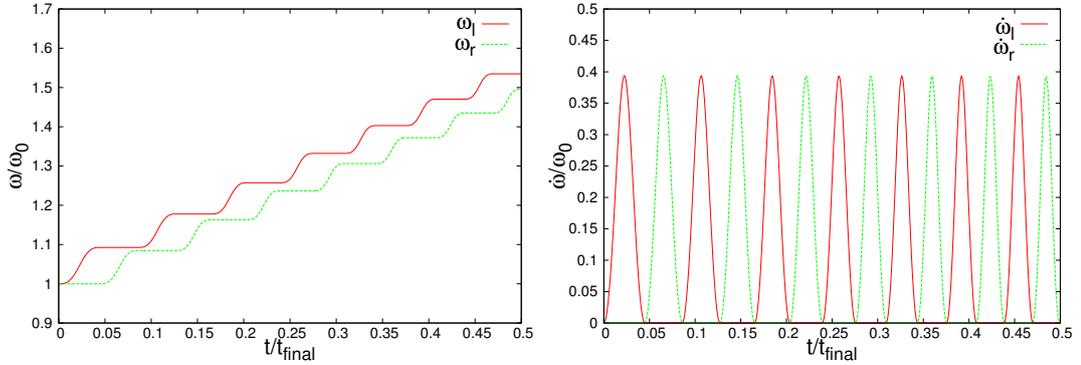


Figure 9: $\omega_1(t)$ and $\omega_r(t)$ of $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$ (left) and their derivations with respect to t (right) for $t_{\text{final}} \approx 11T$

For the other $\omega(q, t)$ like $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ the scaling time τ and ending time t_{final} are approximately the same. But for $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$ these two times are just

3.3 Smoothed $\omega(q, t)$ for the quadratic potential

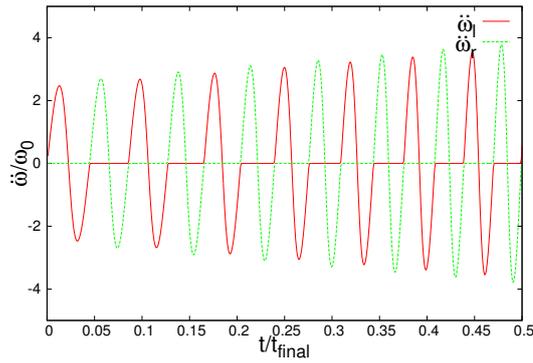


Figure 10: $\ddot{\omega}_1(t)$ and $\ddot{\omega}_r(t)$ of $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$ for $t_{\text{final}} \approx 11T$

approximatively directly proportional to each other. For the adiabatic lines examination it is no difference, but should be mentioned to prevent confusion.

$$\omega_{(\text{pos})}^{(\text{lin})}(q, t) : t_{\text{final}} \approx 1\tau \quad (3.21)$$

$$\omega_{(\text{pos})}^{(\text{smooth})}(q, t) : t_{\text{final}} \approx 10.8\tau \quad (3.22)$$

Now it is time to check the effect of the smooth $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$ on the final phasespaces A_τ as a function of the process duration t_{final} . The diagram in figure 11 shows the five different phasespaces. As before A_1 is constant $\frac{A_0}{2}$ and

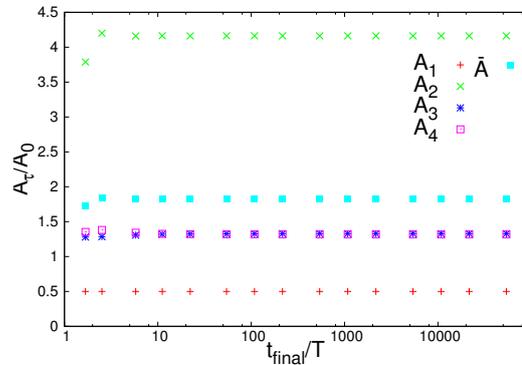


Figure 11: Adiabatic lines of $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$

the other four increased to values greater than A_0 . Interestingly A_2 and \bar{A} reached even higher values for A_τ with $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$ than with $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ or $\omega_{(\text{pos})}^{(\text{quad})}(q, t)$. An explanation for the great increase of A_2 is, that $\omega_1(t)$ and $\omega_r(t)$ increases faster, when $|q_2(t)|$ is at its peak, and increases slow, when $|q_2(t)|$ is

3 Numerical part

small. In this way the energy increase of particle 2 is optimised. Within an accuracy of $0.01A_0$ A_3 and A_4 converge with $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$ at the same value as with $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ or $\omega_{(\text{pos})}^{(\text{quad})}(q, t)$, but this should just be coincidence caused by potentials' similarity.

The intention of this smooth potential is to undermine the equation 2.22. This equation is not full filled, if $\dot{\omega}_1(t)$ and $\dot{\omega}_r(t)$ get faster smaller then $\dot{\omega}_1(t)$ and $\dot{\omega}_r(t)$ in the adiabatic limes, like it is requested in 2.23. In figure 12 you

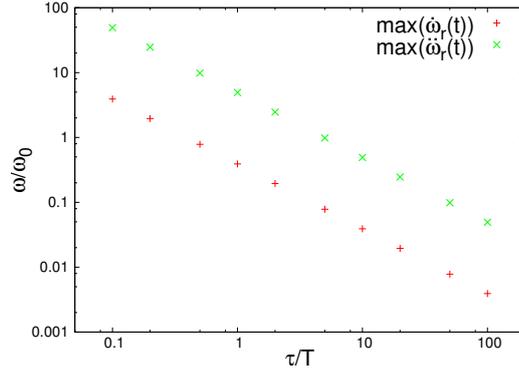


Figure 12: Comparison of $\max |\dot{\omega}_{(\text{pos})r}^{(\text{smooth})}(t)|$ and $\max |\ddot{\omega}_{(\text{pos})r}^{(\text{smooth})}(t)|$

can see $\max |\dot{\omega}_{(\text{pos})r}^{(\text{smooth})}(t)|$ and $\max |\ddot{\omega}_{(\text{pos})r}^{(\text{smooth})}(t)|$ as functions of τ in a double logarithmic scale. The two functions in figure 12 are parallel, that means, that they have the same order of τ . But this does not comply equation 2.23 and the equation 2.22 found by Wells and Siklos could not be disproved.

This example shows also, that it is impossible to create an $\omega_1(t)$ and $\omega_r(t)$, which keeps the energy of one special particle constant, but increases from ω_0 to $2\omega_0$ and $\omega_{1,r}(t)$ is small enough to full fill equation 2.23. For such a $\omega_{1,r}(t)$ it is necessary to increase its derivative from zero to a value proportional to $\frac{1}{\tau}$ in less then an half period, but this requires a second derivative, which is also proportional to $\frac{1}{\tau}$.

3.4 Case of decreasing $\omega(q, t)$

In the last chapters just the case of a from ω_0 to $2\omega_0$ increasing $\omega(q, t)$ is described. Now the complementary case of a from ω_0 to $\frac{1}{2}\omega_0$ decreasing $\omega(q, t)$ is shown. The procedure is the same as in chapter 3.2, so it is not necessary to explain it again. The results for the adiabatic limes are very similar to them from chapter 3.2. The different A_τ converge each to different values and for the last 10 of 14 values, they practically remain constant, so the adiabatic limes should be reached. The big difference is, that A_1 and A_2 switched places.

3.4 Case of decreasing $\omega(q, t)$

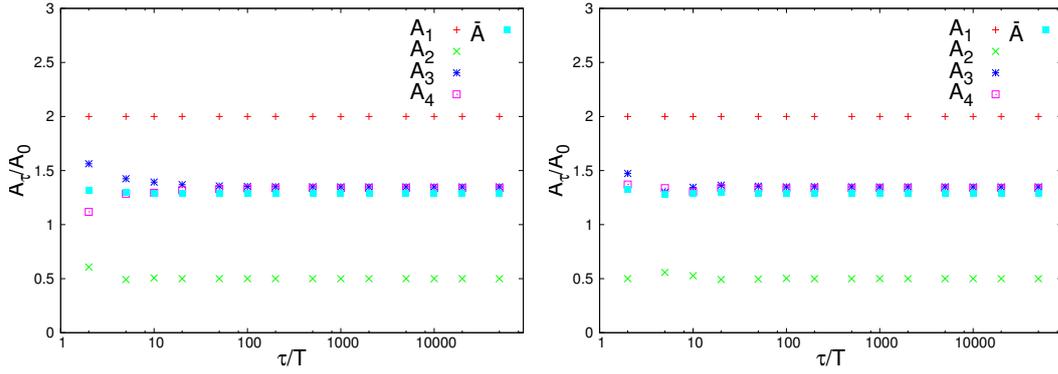


Figure 13: Adiabatic lines of $\omega_{(\text{neg})}^{(\text{lin})}(q, t)$ (left) and $\omega_{(\text{neg})}^{(\text{quad})}(q, t)$ (right)

On the one hand, $\omega_{(\text{neg})}^{(\text{lin})}(q, t)$ and $\omega_{(\text{neg})}^{(\text{quad})}(q, t)$ remain the energy of particle 1 constant. But now $\omega(q, t)$ decreases and that results with $A = \frac{2\pi E}{\omega}$ in a doubled phasespace. On the other hand, particle 2 loses $\frac{3}{4}$ of its energy during the whole process, because it is always on the decreasing branch of the potential. The adiabatic lines of A_3 and A_4 switched also compared to the increasing case. This is not too obvious, because the two values are very similar with $1.338A_0$ and $1.348A_0$. An interesting coincidence is, that in case of increasing and in case of decreasing $\omega(q, t)$ the adiabatic lines of \bar{A} is both times $1.292A_0$.

This coincidence should be a bit more analysed. At the beginning of the process, the thousand particles of \bar{A} have the same initial energy, so forth they all move on the same trajectories in the q - p -plane. These are the red lines in form of an ellipse in figure 14. After the process the particles got different energies and so they move on different trajectories and these trajectories build a region in the q - p -plane. That region are the blue ellipses with holes in the middle in figure 14. This two blue ellipses have the same form, but are rotated with 90° to each other. The trajectories are not homogeneously spread over the blue region, otherwise \bar{A} would be $1.250A_0$ and not $1.292A_0$. Its distributions as a function of the maximal momentum can be seen in figure 15. The height of the distribution diagrams is chosen to keep the shown square footage constant. The distributions ρ are normed, that the integration over the positive momentum space would have the result 1. The comparison of the distribution of \bar{A} in $\omega_{(\text{neg})}^{(\text{lin})}(q, t)$ and $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ solves the mystery, because the distributions are identical apart from the scaling factor 2.

That the same \bar{A} is just a coincidence, shows the example of the $\omega_{(\text{neg})}^{(\text{smooth})}(q, t)$ seen in figure 16. The A_τ values of $\omega_{(\text{neg})}^{(\text{smooth})}(q, t)$ have no accord with them of $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$, though these two potentials are so similar to each other. A quick

3 Numerical part

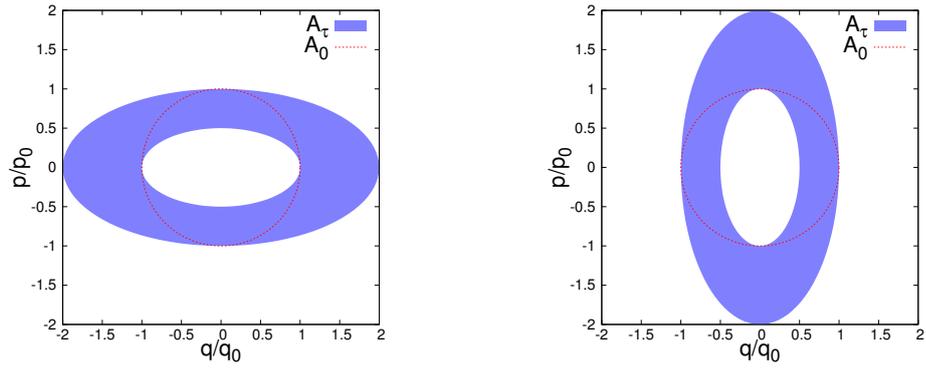


Figure 14: phasespace change of $\omega_{(\text{neg})}^{(\text{lin})}(q, t)$ (left) and $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ (right)

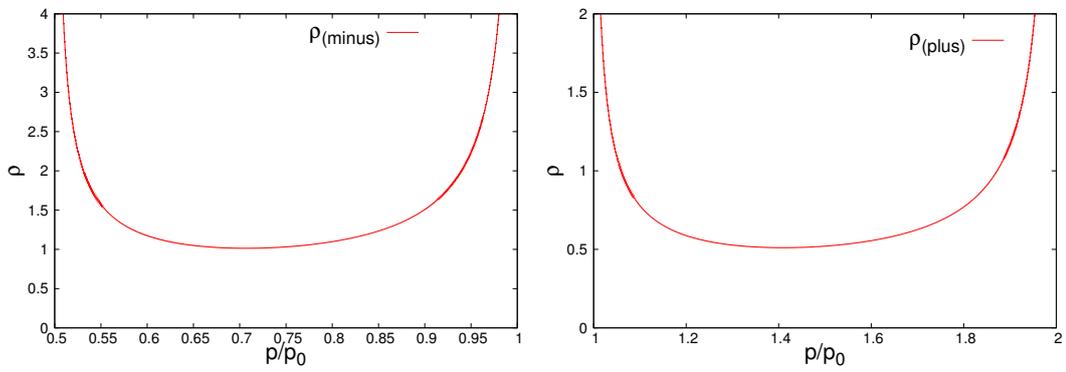


Figure 15: distribution of \bar{A} in $\omega_{(\text{neg})}^{(\text{lin})}(q, t)$ (left) and $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ (right)

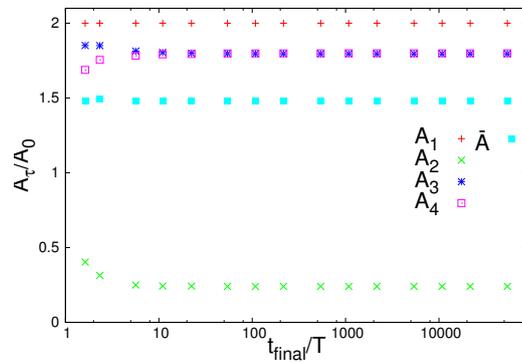


Figure 16: Adiabatic lines of $\omega_{(\text{neg})}^{(\text{smooth})}(q, t)$

view at the evolution from A_0 to A_τ in figure 17 reveals, that for $\omega_{(\text{neg})}^{(\text{smooth})}(q, t)$ and $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$ the trajectories fill different formed spaces in the q - p -plane.

3.5 Results of the biquadratic potential

Not only the trajectories fill different spaces, also the distribution differs as

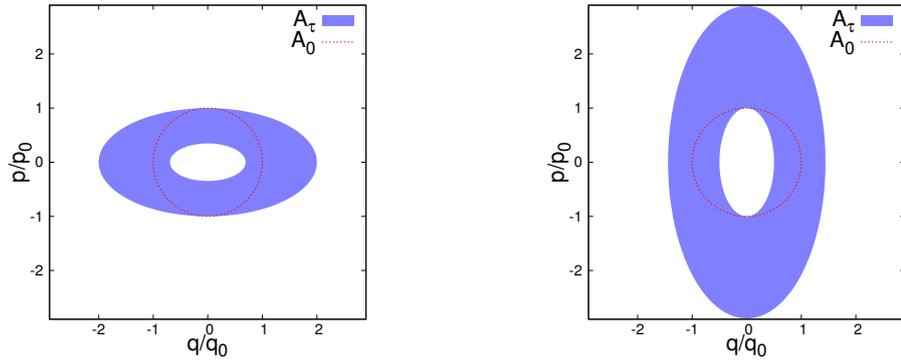


Figure 17: phasespace change of $\omega_{(\text{neg})}^{(\text{smooth})}(q, t)$ (left) and $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$ (right)

seen in figure 18. On the one hand the distribution for $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$ is very symmetric, on the other hand for $\omega_{(\text{neg})}^{(\text{smooth})}(q, t)$ the distribution is significantly higher on the right side, which increases \bar{A} from $\omega_{(\text{neg})}^{(\text{smooth})}(q, t)$, but not to the level of \bar{A} from $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$.

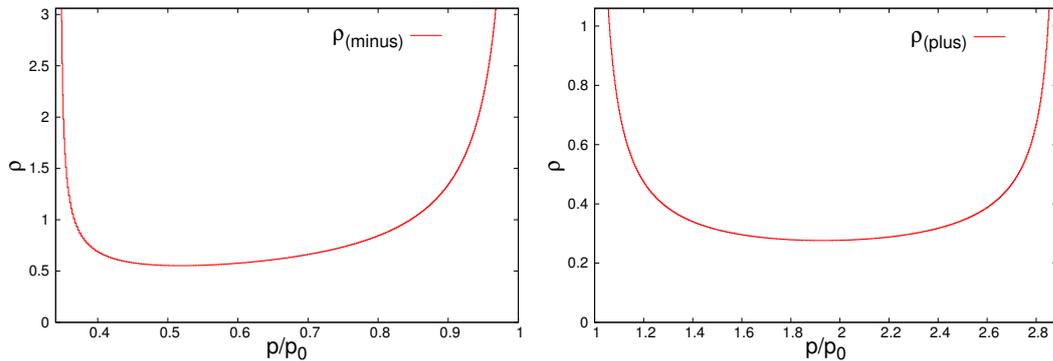


Figure 18: distribution of \bar{A} in $\omega_{(\text{neg})}^{(\text{smooth})}(q, t)$ (left) and $\omega_{(\text{pos})}^{(\text{smooth})}(q, t)$ (right)

3.5 Results of the biquadratic potential

This section is dedicated to the biquadratic potential. One main feature of the biquadratic potential is its energy depending period duration $T(E, \omega)$. On the contrary the period duration of the quadratic potential is only a function of ω . This feature has a great impact to the adiabatic invariance of the phasespace. The evolution to the adiabatic limes for four different $\omega(q, t)$ can be seen in

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figure 19 and 20. Like in the quadratic potential, the $\omega(q, t)$ keep the energy

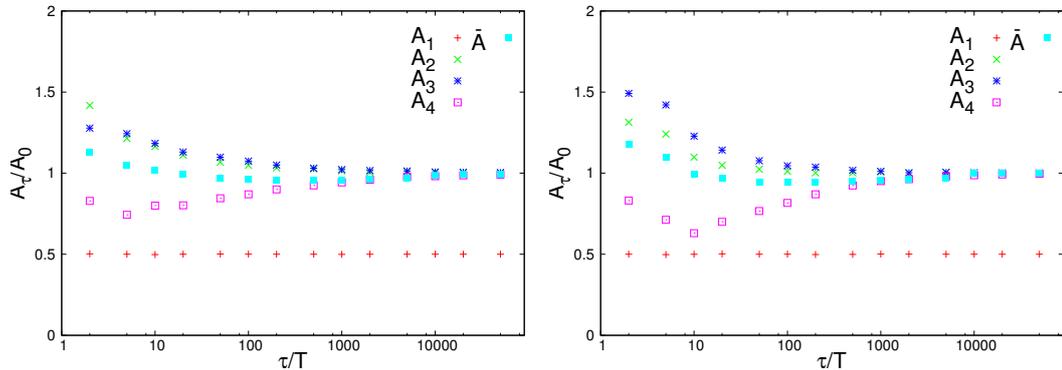


Figure 19: Adiabatic limes of $\omega_{(\text{pos})}^{(\text{lin})}(q, t)$ (left) and $\omega_{(\text{pos})}^{(\text{quad})}(q, t)$ (right) at the biquadratic potential

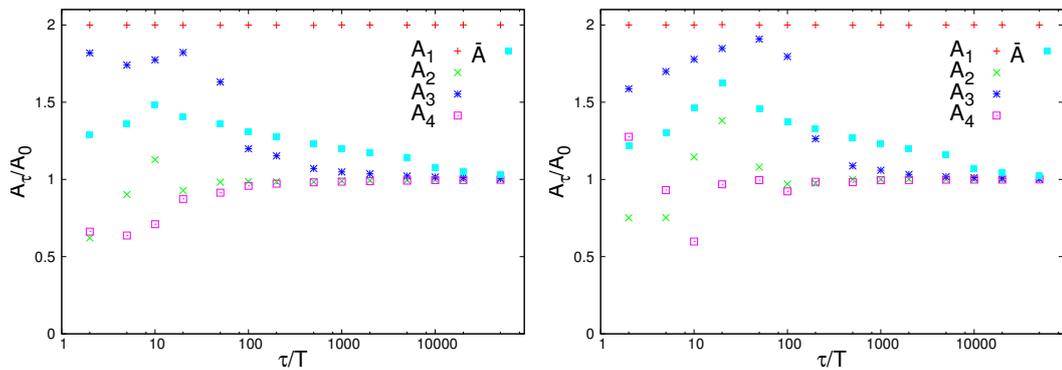


Figure 20: Adiabatic limes of $\omega_{(\text{neg})}^{(\text{lin})}(q, t)$ (left) and $\omega_{(\text{neg})}^{(\text{quad})}(q, t)$ (right) at the biquadratic potential

of particle 1 constant, so that its A_τ is $2A_0$ in case of $\omega_{(\text{pos})}(q, t)$ or $\frac{1}{2}A_0$ in case of $\omega_{(\text{neg})}(q, t)$. But in the biquadratic potential A_τ of the other particles converges to A_0 for big τ . The reason for this is, that the energy of the other particles changes during the process and so their period duration changes. This is not the case for the quadratic potential. Now the period duration of the particles does not accord to the periodic behaviour of the $\omega(q, t)$, which allows the effect of adiabatic invariance of the phasespace. Interestingly the different A_τ converge for the biquadratic potential much slower than for the quadratic potential. A_τ reaches the value of its adiabatic limes with a $\tau = 10000T$ in the biquadratic potential nearly as good as with a $\tau = 10T$ in the quadratic

3.5 Results of the biquadratic potential

potential. So it needs 3 powers of ten more for the biquadratic potential than for the quadratic potential to reach its adiabatic limes.

When you examine the situation accurate, then the phasespace of particle 1 changes in the adiabatic limes and so the phasespace is no adiabatic invariant for the biquadratic potential. Also the difference in the phasespace of particle 1 and e.g. particle 2 disprove the first formula from Wells and Siklos 2.21. On the contrary, the second formula from Wells and Siklos 2.22 works perfectly. The average phasespace is here really an adiabatic invariant.

From a strict mathematical point of view the validation of the formulas is the same for the quadratic and the biquadratic potential. From a more practical point of view the situation is for the quadratic potential different than for the biquadratic potential. On the one hand every calculated phasespace changes in the adiabatic limes in the quadratic potential, in the biquadratic potential on the other hand just the phasespace of particle 1 changes. If there is an error of measurement in the start condition of particle 1, than its phasespace would also converge to A_0 and would be an adiabatic invariant. So practically the phasespace is for the biquadratic potential an adiabatic invariant.

4 Conclusion and outlook

In this bachelor thesis is proven by example, that an arbitrary slowly change in the potential does not result in a constant phasespace.

$$|A(t_{\text{final}}) - A(0)| \neq O(\dot{\lambda}) \quad (4.1)$$

Also, that the phasespace difference of two particles with the same initial energy is in the adiabatic limes constant, is proven to be wrong. That is equivalent to the first formula from Wells and Siklos 2.21.

$$|A_1(t) - A_2(t)| \neq O(\dot{\lambda}) \quad \forall t \in [0, t_{\text{final}}] \quad (4.2)$$

When you take $\ddot{\lambda}$ with in account, like it is done in the second equation from Wells and Siklos 2.22, then all presented examples does not contradict with the adiabatic invariance of the phasespace.

$$|A(t_{\text{final}}) - A(0)| = O(\dot{\lambda}) + \int_0^{t_{\text{final}}} O(\ddot{\lambda}) dt \quad (4.3)$$

Whether this equation applies on the phasespace of every particle, only on an averaged phasespace or not in general, is not answered. But in this thesis is the attempt, to achieve a change in the phasespace besides the adiabatic limes and a small enough $\ddot{\lambda}$, missing. As in chapter 3.3 described, forbids the limitation in $\ddot{\lambda}$ the approach to change the phasespace by using a λ , which changes according to the periodic motion of a particle. So a different approach must be done. Imaginable would be the slow change from one periodic system to another one, e.g. starting with a single well potential and ending with an asymmetric double well potential. Perhaps additional examples does not create a mathematical proof for the adiabatic invariance of the phasespace, but they help to understand its limitations and this helps to develop such a proof.

On the more practical side is shown, that a energy depending period duration helps to keep the phasespace for most particles constant. On the other side, when the period duration is constant for every energy, like in the harmonic oscillator, then a potential, which growth is connected to the period duration, can change the phasespace even in the adiabatic limes.

5 Attachment

5.1 Mathematical Proof: $F_1(q = 0, t) = 0$

Here is proven, that the force $F_1(q = 0, t)$ is zero besides the fact, that the derivation of $\omega(q, t)$ diverges at $q = 0$. The potential $V_1(q, t)$, which causes $F_1(q, t)$, is:

$$V_1(q, t) = \frac{1}{2}\omega^2(q, t)q^2 \quad \text{with} \quad \omega(q, t) = \begin{cases} \omega_r(t) & \text{if } x > 0 \\ \omega_l(t) & \text{if } x \leq 0 \end{cases} \quad (5.1)$$

The definition of $F_1(q = 0, t)$ is:

$$-F_1(q = 0, t) = \lim_{q \rightarrow 0} \left(\frac{\partial V_1(q, t)}{\partial q} \right) \quad (5.2)$$

$$= \lim_{q \rightarrow 0} \left(\lim_{\delta q \rightarrow 0} \frac{\omega^2(q + \delta q, t)(\delta q + q)^2 - \omega^2(q, t)q^2}{2\delta q} \right) \quad (5.3)$$

$$= \lim_{q \rightarrow 0} \left(\lim_{\delta q \rightarrow 0} \frac{\omega^2(q + \delta q, t)(q^2 + 2q\delta q) - \omega^2(q, t)q^2}{2\delta q} \right) \quad (5.4)$$

In the limes of δq and q you have no correlation between δq and q , so you need differ in every possible case. Only when in every case the term converges to zero, then the force $F_1(q = 0, t)$ is zero.

$$1. \text{Case : } \delta q > 0 \text{ and } q > 0 \quad (5.5)$$

$$\lim_{q \rightarrow 0} \left(\lim_{\delta q \rightarrow 0} \frac{\omega_r^2(t)(q^2 + 2q\delta q) - \omega_r^2(t)q^2}{2\delta q} \right) = 0 \quad (5.6)$$

$$2. \text{Case : } \delta q < 0 \text{ and } q < 0 \quad (5.7)$$

$$\lim_{q \rightarrow 0} \left(\lim_{\delta q \rightarrow 0} \frac{\omega_l^2(t)(q^2 + 2q\delta q) - \omega_l^2(t)q^2}{2\delta q} \right) = 0 \quad (5.8)$$

$$3. \text{Case : } \delta q < 0, \quad q > 0 \text{ and } |q| \geq |\delta q| \quad (5.9)$$

$$\lim_{q \rightarrow 0} \left(\lim_{\delta q \rightarrow 0} \frac{\omega_r^2(t)(q^2 + 2q\delta q) - \omega_r^2(t)q^2}{2\delta q} \right) = 0 \quad (5.10)$$

$$4. \text{Case : } \delta q < 0, \quad q > 0 \text{ and } |q| < |\delta q| \quad (5.11)$$

$$\left| \lim_{q \rightarrow 0} \left(\lim_{\delta q \rightarrow 0} \frac{\omega_l^2(t)(q^2 + 2q\delta q) - \omega_r^2(t)q^2}{2\delta q} \right) \right| = \quad (5.12)$$

$$\left| \lim_{q \rightarrow 0} \left(\lim_{\delta q \rightarrow 0} \frac{(\omega_l^2(t) - \omega_r^2(t))q^2}{2\delta q} \right) \right| \leq \quad (5.13)$$

$$\left| \lim_{q \rightarrow 0} \frac{(\omega_l^2(t) - \omega_r^2(t))q^2}{2q} \right| = 0 \quad (5.14)$$

$$5. \text{Case : } \delta q > 0, q < 0 \text{ and } |q| > |\delta q| \quad (5.15)$$

$$\lim_{q \rightarrow 0} \left(\lim_{\delta q \rightarrow 0} \frac{\omega_1^2(t)(q^2 + 2q\delta q) - \omega_1^2(t)q^2}{2\delta q} \right) = 0 \quad (5.16)$$

$$6. \text{Case : } \delta q > 0, q < 0 \text{ and } |q| \leq |\delta q| \quad (5.17)$$

$$\left| \lim_{q \rightarrow 0} \left(\lim_{\delta q \rightarrow 0} \frac{\omega_r^2(t)(q^2 + 2q\delta q) - \omega_1^2(t)q^2}{2\delta q} \right) \right| = \quad (5.18)$$

$$\left| \lim_{q \rightarrow 0} \left(\lim_{\delta q \rightarrow 0} \frac{(\omega_r^2(t) - \omega_1^2(t))q^2}{2\delta q} \right) \right| \leq \quad (5.19)$$

$$\left| \lim_{q \rightarrow 0} \frac{(\omega_r^2(t) - \omega_1^2(t))q^2}{2q} \right| = 0 \quad (5.20)$$

So in every possible case the term converge to zero, which means, that the force $F_1(q = 0, t)$ is really zero.

5.2 mathematical proof: $T = \frac{\partial A}{\partial E}$

This is the derivation of the equation, which calculates the period duration out of the phasespace. A punctual particle is caught in a potential and so it performs a periodic motion with a constant energy E . The Hamiltonian of the system is:

$$H(p, q) = \frac{p^2}{2} + V(q) = E = \text{const.} \quad (5.21)$$

For simplicity the mass of the particle is set to 1. The phasespace A is:

$$A(E) = 2 \int_{q_{\min}(E)}^{q_{\max}(E)} p(q, E) dq = 2 \int_{q_{\min}(E)}^{q_{\max}(E)} \sqrt{2(E - V(q))} dq \quad (5.22)$$

The locations q_{\min} and q_{\max} are the turning points of the periodic motion and so the energy of the particle is at these two points completely transform in potential energy. With the following equation the derivation of the phasespace with respect to the energy can be easily calculated:

$$\partial_y \int_{a(y)}^{b(y)} f(x, y) dx = \int_{a(y)}^{b(y)} \partial_y f(x, y) dx + \frac{\partial b(y)}{\partial y} f(b(y), y) - \frac{\partial a(y)}{\partial y} f(a(y), y) \quad (5.23)$$

The position of the turning points q_{\min} and q_{\max} also depend on the energy, so forth these must also be derivated with respect to E .

$$\frac{\partial A(E)}{\partial E} \stackrel{5.23}{=} 2 \int_{q_{\min}(E)}^{q_{\max}(E)} \partial_E \sqrt{2(E - V(q))} dq +$$

5.2 mathematical proof: $T = \frac{\partial A}{\partial E}$

$$+ \frac{\partial q_{\max}(E)}{\partial E} \sqrt{2(E - V(q_{\max}))} - \frac{\partial q_{\min}(E)}{\partial E} \sqrt{2(E - V(q_{\min}))} \quad (5.24)$$

$$= 2 \int_{q_{\min}(E)}^{q_{\max}(E)} \frac{1}{\sqrt{2(E - V(q))}} dq + \frac{\partial q_{\max}(E)}{\partial E} \sqrt{2(E - E)} - \frac{\partial q_{\min}(E)}{\partial E} \sqrt{2(E - E)} \quad (5.25)$$

Here is the momentum $p = \frac{dq}{dt}$. With this the integration over the position q can be replaced by a integration over time t : $\frac{dq}{p} = dt$

$$= 2 \int_{q_{\min}(E)}^{q_{\max}(E)} \frac{1}{p} dq = 2 \int_{t(q_{\min})}^{t(q_{\max})} dt = T \quad (5.26)$$

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Declaration

I hereby declare that I have authored this thesis independently, that I have not used other than the declared sources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

Augsburg, October 6, 2013

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